Hyper-operations as a Tool for Science and Engineering

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The analysis and simulation of physical and engineering phenomena often requires discontinuous functions. Particular "*ad-hoc*" models are used including step functions, infinite pulse signals, or abstract concept like "dot events". Despite the successful approximations of such models, an accurate mathematical description of such signals is normally lacking. Moreover, the evolution of computer science requires the storage and handling of extremely large and extremely small numbers, with a permanent danger of "computing overflow".

The present Poster has two objectives:

- To show a possible extension of existing mathematical tools for describing discontinuous signals, using a new arithmetical operation called "*zeration*";
- To propose a new number notation, using of a very powerful compact operation, the iterated exponentiation, called "*tetration*" or "*superpower*", with some extra complementary tools.

These objectives are reachable via the definition of two "hyper-operations", both justified by Ackermann's Function:

$$A(s,n) = \{z \boxed{s} (n+3)\} - 3$$, where $\boxed{s}$ indicates the rank s hyper-operation.

From this definition, the new *zeration* operation (ο, s = 0) immediately follows, with a rank lower than *addition* (+, s = 1) and showing the following properties:

$$a \circ b = \{a+1 ,\ \forall a > b \ ;\ b+1 ,\ \forall b > a \ ;\ a+2 = b+2 ,\ \forall a = b \ ;\ a ,\ \forall b = (-\infty) \ ;\ b ,\ \forall a = (-\infty)\}$$

with neutral unit: $\varepsilon = -\infty$

Zeration

Zeration, for: $a = n$ ($n \in$ N) and $b = 0$, gives $a \circ b = n \circ 0 = n+1$ and the result coincides with the "successor of n". Its inverse operation (called *deltation*) generates a new class of numbers (*delta numbers*), the properties of which are under study. For example, it can be proved that:

$$a^{\Delta b} = 0 - \left(a^b\right) = -a^b \qquad \text{where: } b \in \mathrm{R}$$

similar to $a^{-b} = 1 \ / \ a^b = \ : a^b$ therefore, we can put:

$$a^{\Delta b} = -\,a^b$$

Delta numbers appear to be a new class D_0 of numbers, obtained as logarithms of negative numbers (normally represented as multi-value complex numbers):

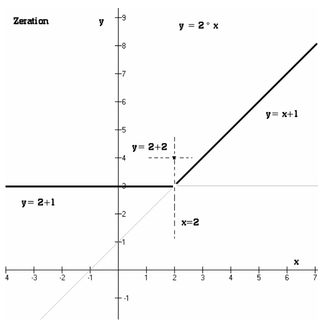
$$\Delta b \in D_0 \quad \text{with:} \quad D_0 = \{\log(-a), a > 0\}$$

It can be shown that *zeration* allows the exact definition of functions such as **step**, **sign**, **max**, **min**, etc. For example, **abs(x)** function can be written as follows:

$$|x| = (1 - (-x \circ 0)) \cdot ((x \circ 0) - 2) + ((x \circ 0) - 1) \cdot (2 - (-x \circ 0))$$

Moreover, *zeration* allows a new approach to the non-standard "actual infinite". Starting from Ackermann's function, the *tetration* operation (*super-power*, *power-tower*, *tower*) and its two inverse operations (*super-root* and *super-logarithm*) can also be defined. Their study shows interconnections with numbers obtained from deltation and from other inverse hyper-operations with ranks lower than *addition* (s ≤ 0).

Graph of $y = 2 \circ x$:



Tetration

By means of *tetration* (non commutative) and of one of its two inverses (the super-logarithm), a new number notation format is proposed, structurally similar to the "floating point" notation, but using *tetration* instead of *exponentiation*. In fact, let us consider the following formulas:

$$x = b \cdot (b \cdot (b \cdot (b \cdot (\ldots (b \cdot p) \ldots)))) = p \cdot b^n \qquad 1 \le p < b$$
...... n times ....

$$x = b \,\hat{}\, (b \,\hat{}\, (b \,\hat{}\, (b \,\hat{}\, (\ldots (b \,\hat{}\, p) \ldots)))) = p * {}^n b \qquad 1 \le p < b$$
...... n times ....

Therefore, we can write: $$x = p * {}^n b$$

The first expression, for b = 10, shows the structure of the scientific "floating point" number notation, where the parentheses are not necessary (multiplication is commutative and associative). The second expression shows the new proposed "tetrational" number notation, where the parentheses are indispensable, because exponentiation is neither commutative nor associative. Number p is the last exponent (first exponentiation to be executed), called the "tower extension" and it represents the precision of the tetrational notation. Operator (*) is called the extension operator and ${}^n b$ (b-tetra-n) is the tetration operation itself. In particular, we have:

$$x = p * {}^n b = {}^{(n + \mathrm{slog}_b\, p)} b \qquad \text{where: } 1 \le p < b$$

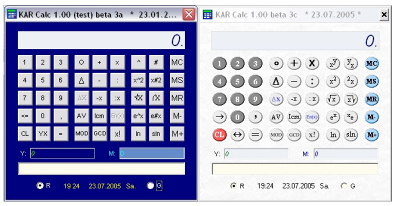
and:
p : tower extension, tetrational precision, tetrational *significant*,
b : tetrational *base*,
n : *super-exponent*, tetrational order of magnitude.

Number $q = \mathrm{slog}_b\, p$ (with $0 \le q < 1$) is the "*super-mantissa*" of $\mathrm{slog}_b\, x$. This new format for number notation, suitable for very large and very small numbers, has been generalized, at various hyper-operations levels (**RRH**©, the Rubtsov-Romerio number notation hyper-format, © 2005-2010, SIAE Deposit no. 0501678, Rome, 21-04-2005).

With the above-mentioned implementation (**RRH**©, rank s = 4), it is possible representing very large numbers, as well as those obtained by inverse operations with ranks lower than addition. Its development is based on the study of homomorphic ω -correspondences between mathematical objects.

A practical machine storage format (rank s=4) has been implemented and adapted to existing standards. It allows the storage of all the **RRH**© components, together with of other number characteristics (e.g.: Δ, minus and reciprocal signs, etc.).

A prototype software for an hyper-calculator including the hyper-operations of ranks s = 0, 1, 2, 3 and 4 has been developed.



A prototype of a number notation hyper-converter, from the scientific "floating point" to the tetrational notation (with bases 2, e and 10) and vice-versa, has also been implemented.